

# Momentum-space Lippmann-Schwinger-Equation, Fourier-transform with Gauss-Expansion-Method

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In these notes we construct the momentum-space potentials from configuration-space using for the Fourier-transformation the Gaussian-Expansion-Method (GEM) [1]. This has the advantage that the Fourier-Bessel integrals can be performed analytically, avoiding possible problems with oscillations in the Bessel-functions for large  $r$ , in particular for  $p_f \neq p_i$ . The mass parameters in the exponentials of the Gaussian-base functions are fixed using the geometric progression recipe of Hiyama-Kamimura. The fitting of the expansion coefficients is linearly.

Application for nucleon-nucleon is given in detail for the recent extended soft-core model ESC08c. The NN phase shifts obtained by solving the Lippmann-Schwinger equations agree very well with those obtained in configuration space.

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## I. INTRODUCTION

In these notes the soft-core momentum-space potentials are constructed from configuration-space using for the Fourier-transformation the Gaussian-Expansion-Method (GEM) [1]. With gaussian form factors this gives a most natural practical presentation in momentum space. This can be seen, using the Schwinger representation of the meson propagator, as follows

$$\begin{aligned}\tilde{V}_{i,\alpha}(\mathbf{k}^2) &= g^2 \frac{e^{-\mathbf{k}^2/\Lambda^2}}{\mathbf{k}^2 + m^2} = g^2 \int_0^\infty d\alpha e^{-\alpha(\mathbf{k}^2 + m^2)} e^{-\mathbf{k}^2/\Lambda^2} \\ &= \bar{g}^2 \int_{1/\Lambda^2}^\infty d\gamma e^{-\gamma m^2} e^{-\gamma \mathbf{k}^2}, \quad \bar{g}^2 = g^2 \exp\left(\frac{m^2}{\Lambda^2}\right), \\ V_{i,\alpha}(r) &= \frac{\bar{g}^2}{2\pi\sqrt{\pi}} \int_0^{2\Lambda} d\mu e^{-\mu^2 r^2} \approx \sum_k w_{\alpha,k}^{(i)} e^{-\mu_k^2 r^2},\end{aligned}\tag{1.1}$$

where the transition to the approximate form can be realized by using appropriate quadratures. The label  $\alpha$  represents the set  $(g, m, \Lambda)$ , and the real potentials are sums (integrals) over the set  $\{\alpha\}$  for each type  $i$  of potential. The types considered are the central, spin-spin, tensor, spin-orbit, and quadratic-spin-orbit potentials, i.e.  $i = C, \sigma, T, SO, Q$ .

The necessary Fourier-Bessel integrals can be performed analytically, avoiding possible problems with the oscillations in the Bessel-functions for large  $r$ , in particular for  $p_f \neq p_i$ .

The mass parameters in the exponentials of the Gaussian-base functions are fixed using the geometric progression recipe of Hiyama-Kamimura. The fitting of the remaining parameters, the expansion coefficients  $w_{\alpha,k}^{(i)}$ , is linearly, requiring only a couple of steps to reach the optimal parameters. In addition to the potential values at a set of distances  $\{r_n, n = 1, N\}$ , we fit the volume integrals.

The soft-core potentials can be evaluated directly in momentum space using the analytic forms of the potentials, see e.g. [2]. However, in the case of the ESC-model also the momentum space potentials require the execution of various types of numerical integrals for each set  $(p_i, p_j)$ , where  $i$  and  $j$  run over the used mesh-points of the quadrature which is used for solving the Lippmann-Schwinger equation, and is rather time consuming. In the method described here we use the configuration potentials, which for ESC are energy independent, as input and fix the GEM-expansion coefficients for each channel, e.g. in the NN-case for  $I=0,1$ , or pp, np and nn. From these the momentum space potentials can be computed very efficiently and fast.

In the case of hyperon-nucleon and hyperon-hyperon this method can readily be generalized. Then, the coefficients  $w_i$  become matrices in channel space. For example for the coupled channels  $\Lambda N, \Sigma N$  one has for isospin  $I=1/2$  for each coefficient a  $2 \times 2$ -matrix.

The content of these notes is as follows. In section II and III the Lippmann-Schwinger equation, the relation to the partial wave K-matrix, phase shifts, and the used units are given. In section IV the Fourier transform for a general potential form for the central, tensor, spin-orbit, and quadratic-spin-orbit potential is derived in detail. In section V The form factor is included in particular the gaussian form factor which is essential for the Nijmegen soft-core potentials. The Gauss-Bessel radial integrals are evaluated in section VI. In section VII the application to the ESC-potentials is given. Here we introduce the explicit form of the used GEM expansion. In section VIII contains the results for the recent ESC08c potential [3, 4]. We demonstrate the method

by giving the phases obtained by solving the Lippmann-Schwinger equation by either the Kowalski-Noyes [5] or the Haftel-Tabakin [6] method. Both methods give essentially the same results. Finally, in section VIII these notes are closed by a brief discussion and concluding remarks. In Appendix A the general expressions for the Gauss-Bessel integral  $I_{n,n+2}$  is checked by a second method of evaluation. In Appendix B the general expressions for the Gauss-Bessel integrals are checked by an explicit evaluation for  $I_{0,0}$  and  $I_{0,2}$ .

## II. LIPPMANN-SCHWINGER EQUATION, NR-NORMALIZATION

With the non-relativistic normalization of the two-particle states

$$(\mathbf{p}', s'_1; \mathbf{p}'_2, s'_2 | \mathbf{p}_1, s_1; \mathbf{p}_2, s_2) = (2\pi)^6 \delta(\mathbf{p}'_1 - \mathbf{p}_1) \delta(\mathbf{p}'_2 - \mathbf{p}_2) \delta_{s'_1, s_1} \delta_{s'_2, s_2}, \quad (2.1)$$

the Lippmann-Schwinger equation LSE) reads

$$(3, 4 | T | 1, 2) = (3, 4 | V | 1, 2) + \sum_n \int \frac{d^3 p_n}{(2\pi\hbar)^3} (3, 4 | V | n_1, n_2) \frac{2\mu_{n_1, n_2}}{p_E^2 - p_n^2 + i\epsilon} (n_1, n_2 | T | 1, 2). \quad (2.2)$$

The partial-wave LSE, restricting ourselves to single channel elastic scattering, in the CM-system reads

$$(p_f | T^J | p_i) = (p_f | V^J | p_i) + \frac{1}{2\pi^2} \int_0^\infty dp_n p_n^2 (p_f | V^J | p_n) \frac{2\mu_{red}}{p_E^2 - p_n^2 + i\epsilon} (p_n | T^J | p_i) \quad (2.3)$$

Now, the dimensions are  $[V^J] = [T^J] = [\text{MeV}]^{-2}$ , for units where  $\hbar = c = 1$ .

Likewise for the K-matrix the partial wave LSE reads

$$(p_f | K^J | p_i) = (p_f | V^J | p_i) + \frac{1}{2\pi^2} P \int_0^\infty dp_n p_n^2 (p_f | V^J | p_n) \frac{2\mu_{red}}{p_E^2 - p_n^2} (p_n | K^J | p_i) \quad (2.4)$$

Transforming

$$\tilde{K}^J \equiv \frac{1}{4\pi} \sqrt{2\mu_{red}} K^J \sqrt{2\mu_{red}} \quad (2.5)$$

leads, in the obvious notation  $\tilde{K}^J(p_f, p_i) \equiv (p_f | \tilde{K}^J | p_i)$ , to the LSE

$$\tilde{K}^J(p_f, p_i) = \tilde{V}^J(p_f, p_i) + \frac{2}{\pi} P \int_0^\infty dp_n p_n^2 \tilde{V}^J(p_f, p_n) \frac{1}{p_E^2 - p_n^2} \tilde{K}^J(p_n, p_i) \quad (2.6)$$

## III. K-MATRIX AND PHASE-SHIFTS

The differential X-section is given by

$$\frac{d\sigma}{d\Omega} = |F|^2, \quad F = -\frac{2\mu_{red}c^2}{4\pi(\hbar c)^2} T, \quad (3.1)$$

which means  $[F] = [\text{fm}]$ , and using  $[\hbar c] = [\text{MeV}][\text{fm}]$  gives  $[T] = [V] = [\text{MeV}][\text{fm}]^3$ .

In general units the transformation (2.5) reads

$$\tilde{K}^J \equiv \frac{1}{4\pi} \frac{\sqrt{2\mu_{red}c^2}}{(\hbar c)} K^J \frac{\sqrt{2\mu_{red}c^2}}{(\hbar c)}. \quad (3.2)$$

Then, in general units The LSE for the K-amplitude is [7]

$$\begin{aligned} \tilde{K}^J(p_f, p_i) &= \tilde{V}^J(p_f, p_i) + \frac{2}{\pi} P \int_0^\infty \frac{d(p_n c)}{(\hbar c)} \tilde{V}^J(p_f, p_n) \frac{(p_n c)^2}{(p_E c)^2 - (p_n c)^2} \tilde{K}^J(p_n, p_i) \\ &\equiv \tilde{V}^J(p_f, p_i) + P \int_0^\infty d(p_n c) \tilde{V}^J(p_f, p_n) \tilde{g}_E(p_n) \tilde{K}^J(p_n, p_i), \end{aligned} \quad (3.3)$$

with

$$\tilde{g}_E(\mathbf{p}_n) = +\frac{2}{\pi}(\hbar c)^{-1} \frac{(\mathbf{p}_n c)^2}{(\mathbf{p}_E^2 c) - (\mathbf{p}_n c)^2}, \quad \tan \delta = -\frac{pc}{\hbar c} \tilde{K}(\mathbf{p}_E, \mathbf{p}_E). \quad (3.4)$$

Comparing (3.1) and the transformation (3.2), it follows that for the transformed T-matrix we have that  $\tilde{T} = -F$ . For the partial waves we have

$$F^J(p) = -\tilde{T}^J = -\frac{1}{4\pi} \frac{\sqrt{2m_{red}c^2}}{(\hbar c)} T^J \frac{\sqrt{2m_{red}c^2}}{(\hbar c)}, \quad (3.5)$$

which implies for the elastic phase-shift

$$\tan \delta_J = -\frac{pc}{\hbar c} \tilde{K}^J = -\frac{2m_{red}c^2}{4\pi(\hbar c)^2} \frac{pc}{\hbar c} K^J. \quad (3.6)$$

#### IV. FOURIER-TRANSFORM CONFIGURATION VICE-VERSA MOMENTUM SPACE

For establishing the details this is more complicated for the tensor-, spin-orbit-, and quadratic-spin-orbit-potential than for the central-potential. Therefore we give more explicit details of the derivation of the formulas for these potentials in momentum-space by making a Fourier-transform from configuration space.

##### A. Configuration- and Momentum-space States

The normalization of the states and wave functions we use [8]

$$\langle \mathbf{r}' | \mathbf{r} \rangle = \delta^3(\mathbf{r}' - \mathbf{r}), \quad (4.1)$$

$$\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}), \quad (4.2)$$

$$\langle \mathbf{r} | \mathbf{p} \rangle = \exp(i\mathbf{p} \cdot \mathbf{r}). \quad (4.3)$$

For particles with momentum  $\mathbf{p}$  and orbital angular momentum  $\mathbf{l}$  the state, apart from the spin part, is given by

$$|\mathbf{p}l\mathbf{m}\rangle = \int d\Omega_{\hat{\mathbf{p}}} Y_{lm}(\hat{\mathbf{p}}) |\mathbf{p}\rangle, \quad |\mathbf{p}\rangle = \sum_{lm} Y_{lm}^*(\hat{\mathbf{p}}) |\mathbf{p}l\mathbf{m}\rangle. \quad (4.4)$$

here,  $|\mathbf{p}l\mathbf{m}\rangle$  is an eigenstate of the angular momentum operator  $\mathbf{L}$ . From these introductory definitions and normalizations we obtain the following matrix elements

$$\begin{aligned} \langle \mathbf{r} | \mathbf{p}l\mathbf{m} \rangle &= \int d\Omega_{\hat{\mathbf{p}}} Y_{lm}(\hat{\mathbf{p}}) \cdot 4\pi \sum_{l'm'} i^{l'} j_{l'}(kr) Y_{l'm'}^*(\hat{\mathbf{p}}) Y_{l'm'}(\hat{\mathbf{r}}) \\ &= 4\pi i^{l'} j_{l'}(kr) Y_{lm}(\hat{\mathbf{r}}), \end{aligned} \quad (4.5)$$

$$\begin{aligned} \langle \mathbf{p}'l'm' | \mathbf{p} \rangle &= \int d\Omega_{\hat{\mathbf{p}}'} Y_{l'm'}^*(\hat{\mathbf{p}}') (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}) \\ &= (2\pi)^3 \frac{\delta(\mathbf{p}' - \mathbf{p})}{p^2} Y_{l'm'}^*(\hat{\mathbf{p}}'). \end{aligned} \quad (4.6)$$

The extension of this last matrix element including spin is straightforward. One has

$$\langle \mathbf{p}'s' | \mathbf{p}, LSJM \rangle = (2\pi)^3 \frac{\delta(\mathbf{p}' - \mathbf{p})}{p^2} \mathcal{Y}_{LSJM}(\hat{\mathbf{p}}', s'), \quad (4.7)$$

where

$$\mathcal{Y}_{LSJM}(\hat{\mathbf{p}}', s') = \sum_{m,\mu} C_{M m \mu}^{L S} Y_{Lm}(\hat{\mathbf{p}}') \chi_{\mu}^{(S)}(s'). \quad (4.8)$$

Here,  $s'$  denotes a particular spin variable of the BB-system, e.g. the helicity, or the transversal spin component, or the projection along the z-axis. For the latter, which will be used here,  $s' = \mu$ .

### B. Fourier-Transform Central-Potential

The partial-wave matrix elements in momentum space can be related to those in configuration space as follows

$$\begin{aligned} \langle p_f L_f m_f | \tilde{V}_C(\mathbf{k}^2) | p_i L_i m_i \rangle &= \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \langle p_f L_f m_f | \mathbf{p}' \rangle \langle \mathbf{p}' | V_C | \mathbf{p} \rangle \langle \mathbf{p} | p_i L_i m_i \rangle = \\ &= \int d\Omega_{\hat{\mathbf{p}}_f} \int d\Omega_{\hat{\mathbf{p}}_i} Y_{L_f m_f}^*(\hat{\mathbf{p}}_f) Y_{L_i m_i}(\hat{\mathbf{p}}_i) \cdot \left[ \int d^3 r e^{-i\mathbf{p}_f \cdot \mathbf{r}} V_C(r) e^{+i\mathbf{p}_i \cdot \mathbf{r}} \right]. \end{aligned} \quad (4.9)$$

From Bauer's formula

$$\exp(i\mathbf{p} \cdot \mathbf{r}) = 4\pi \sum_{l,m} i^l j_l(pr) Y_{lm}^*(\hat{\mathbf{p}}) Y_{lm}(\hat{\mathbf{r}}) \quad (4.10)$$

we have that

$$\begin{aligned} \int d\Omega_{\hat{\mathbf{p}}} Y_{L_i m_i}(\hat{\mathbf{p}}) e^{+i\mathbf{p} \cdot \mathbf{r}} &= 4\pi i^{-L_i} Y_{L_i m_i}(\hat{\mathbf{r}}) j_{L_i}(pr), \\ \int d\Omega_{\hat{\mathbf{p}}'} Y_{L_f m_f}^*(\hat{\mathbf{p}}') e^{-i\mathbf{p}' \cdot \mathbf{r}} &= 4\pi i^{+L_f} Y_{L_f m_f}^*(\hat{\mathbf{r}}) j_{L_f}(pr). \end{aligned}$$

Substitution into (4.9) leads finally to the desired formula

$$\begin{aligned} \langle p_f L_f m_f | \tilde{V}_C(\mathbf{k}^2) | p_i L_i m_i \rangle &= +(4\pi)^2 i^{L_f - L_i} \cdot \\ &\times \int d^3 r [Y_{L_f m_f}^*(\hat{\mathbf{r}}) Y_{L_i m_i}(\hat{\mathbf{r}})] \cdot [j_{L_f}(p_f r) V_C(r) j_{L_i}(p_i r)] \Rightarrow \\ &+ (4\pi)^2 \delta_{L_f, L_i} \delta_{m_f, m_i} \int_0^\infty r^2 dr j_{L_f}(p_f r) V_C(r) j_{L_i}(p_i r). \end{aligned} \quad (4.11)$$

### C. Fourier-Transform Tensor-Potential

The partial-wave matrix elements in momentum space can be related, analogous to (4.9) etc., to those in configuration space as follows

$$\begin{aligned} \langle p_f L_f m_f | \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_3(\mathbf{k}^2) | p_i L_i m_i \rangle &= \\ \int \frac{d^3 p'}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \langle p_f L_f m_f | \mathbf{p}' \rangle \langle \mathbf{p}' | \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_3(\mathbf{k}^2) | \mathbf{p} \rangle \langle \mathbf{p} | p_i L_i m_i \rangle \Rightarrow \\ &- \int d\Omega_{\hat{\mathbf{p}}_f} \int d\Omega_{\hat{\mathbf{p}}_i} Y_{L_f m_f}^*(\hat{\mathbf{p}}_f) Y_{L_i m_i}(\hat{\mathbf{p}}_i) \cdot \\ &\times \left[ \int d^3 r e^{+i\mathbf{k} \cdot \mathbf{r}} \left( \boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla} \boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla} - \frac{1}{3} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \nabla^2 \right) V_3(r) \right] = \\ &- \int d\Omega_{\hat{\mathbf{p}}_f} \int d\Omega_{\hat{\mathbf{p}}_i} Y_{L_f m_f}^*(\hat{\mathbf{p}}_f) Y_{L_i m_i}(\hat{\mathbf{p}}_i) \cdot \\ &\times \left[ \int d^3 r e^{+i\mathbf{k} \cdot \mathbf{r}} \left( \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \frac{1}{3} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \left( \frac{1}{r} \frac{d}{dr} - \frac{d^2}{dr^2} \right) V_3(r) \right] \equiv \\ &- \int d\Omega_{\hat{\mathbf{p}}_f} \int d\Omega_{\hat{\mathbf{p}}_i} Y_{L_f m_f}^*(\hat{\mathbf{p}}_f) Y_{L_i m_i}(\hat{\mathbf{p}}_i) \cdot \left[ \int d^3 r e^{-i\mathbf{p}' \cdot \mathbf{r}} S_{12}(\hat{\mathbf{r}}) V_T(r) e^{+i\mathbf{p} \cdot \mathbf{r}} \right] | p_i L_i m_i \rangle. \end{aligned} \quad (4.12)$$

Here, we used partial integration in order to transfer the derivatives to the basic function  $V_0(r)$ , and

$$S_{12}(\hat{\mathbf{r}}) = 3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (4.13)$$

$$V_T(r) = -\frac{1}{3} \left( \frac{1}{r} \frac{d}{dr} - \frac{d^2}{dr^2} \right) V_3(r). \quad (4.14)$$

From Bauer's formula (4.10) we have that

$$\begin{aligned} \int d\Omega_{\hat{\mathbf{p}}} Y_{L_i m_i}(\hat{\mathbf{p}}) e^{+i\mathbf{p} \cdot \mathbf{r}} &= 4\pi i^{-L_i} Y_{L_i m_i}(\hat{\mathbf{r}}) j_{L_i}(pr), \\ \int d\Omega_{\hat{\mathbf{p}}'} Y_{L_f m_f}^*(\hat{\mathbf{p}}') e^{-i\mathbf{p}' \cdot \mathbf{r}} &= 4\pi i^{+L_f} Y_{L_f m_f}^*(\hat{\mathbf{r}}) j_{L_f}(pr). \end{aligned}$$

Substitution into (4.12) leads to the desired formula

$$\begin{aligned} \langle p_f L_f m_f | \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_3(\mathbf{k}^2) | p_i L_i m_i \rangle &= \\ - (4\pi)^2 i^{L_f - L_i} \int d^3 r [Y_{L_f m_f}^*(\hat{\mathbf{r}}) S_{12}(\hat{\mathbf{r}}) Y_{L_i m_i}(\hat{\mathbf{r}})] \cdot [j_{L_f}(p_f r) V_T(r) j_{L_i}(p_i r)] &\Rightarrow \\ - (4\pi)^2 i^{L_f - L_i} \delta_{M_f, M_i} (J M_f, S_f L_f || S_{12} || J M_i, L_i S_i) \cdot & \\ \times \int_0^\infty r^2 dr [j_{L_f}(p_f r) V_T(r) j_{L_i}(p_i r)], & \end{aligned} \quad (4.15)$$

which relates the configuration space tensor-potential to the momentum space one. Here,

$$\int d\Omega_{\hat{\mathbf{r}}} [\mathcal{Y}_{M_f, L_f S_f}^*(\hat{\mathbf{r}}) S_{12}(\hat{\mathbf{r}}) \mathcal{Y}_{M_i, L_i S_i}(\hat{\mathbf{r}})] \equiv (J, L_f || S_{12} || J, L_i) \delta_{S_f, S_i} \delta_{M_f, M_i}, \quad (4.16)$$

with only for total spin  $S_f = S_i = 1$  non-zero matrix elements. We have two cases:

(i) triplet-uncoupled:  $L_f = L_i = J$ :  $\langle S_{12} \rangle = (J || S_{12} || J) = 2$ .

(ii) triplet-coupled:  $L_f = J \pm 1$  and  $L_i = J \pm 1$ :  $L_f$  and  $L_i$  are  $L \pm 1$ ,

$$\langle S_{12} \rangle = (L_f || S_{12} || L_i) = \frac{1}{2J+1} \begin{pmatrix} -2J+2 & 6\sqrt{J(J+1)} \\ 6\sqrt{J(J+1)} & -2J-4 \end{pmatrix} \quad (4.17)$$

#### D. Fourier-Transform Spin-Orbit-Potential

The partial-wave matrix elements in momentum space can be related to those in configuration space as follows

$$\begin{aligned} \langle p_f L_f m_f | \frac{i}{2} (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{n} \tilde{V}_0(\mathbf{k}^2) | p_i L_i m_i \rangle &= \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \cdot \\ \times \langle p_f, L_f m_f | \mathbf{p}' \rangle \langle \mathbf{p}' | \frac{i}{2} (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathbf{q} \times \mathbf{k}) \tilde{V}_0(\mathbf{k}^2) | \mathbf{p} \rangle \langle \mathbf{p} | p_i L_i m_i \rangle &\Rightarrow \\ - \int d\Omega_{\hat{\mathbf{p}}'} \int d\Omega_{\hat{\mathbf{p}}} Y_{L_f m_f}^*(\hat{\mathbf{p}}') Y_{L_i m_i}(\hat{\mathbf{p}}) \cdot & \\ \times \left[ \int d^3 r e^{+i\mathbf{k} \cdot \mathbf{r}} \frac{1}{2} (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathbf{q} \times \boldsymbol{\nabla}) V_0(r) \right] &= \\ - \int d\Omega_{\hat{\mathbf{p}}'} \int d\Omega_{\hat{\mathbf{p}}} Y_{L_f m_f}^*(\hat{\mathbf{p}}') Y_{L_i m_i}(\hat{\mathbf{p}}) \cdot \left[ \int d^3 r e^{+i\mathbf{k} \cdot \mathbf{r}} \mathbf{S} \cdot (\mathbf{r} \times \mathbf{q}) V_{SO}(r) \right] & \\ - \int d\Omega_{\hat{\mathbf{p}}'} \int d\Omega_{\hat{\mathbf{p}}} Y_{L_f m_f}^*(\hat{\mathbf{p}}') Y_{L_i m_i}(\hat{\mathbf{p}}) \cdot \left[ \int d^3 r e^{+i\mathbf{p}' \cdot \mathbf{r}} \mathbf{L} \cdot \mathbf{S} V_{SO}(r) e^{-i\mathbf{p} \cdot \mathbf{r}} \right], & \end{aligned} \quad (4.18)$$

where

$$\mathbf{L} = \frac{1}{2} \mathbf{r} \times (\mathbf{p}' + \mathbf{p}), \text{ and } V_{SO}(r) = -\frac{1}{r} \frac{d}{dr} V_0(r). \quad (4.19)$$

**Note:** Relation partial wave integrals  $V_{SO}$  and  $V_0$ . The partial wave projection of the spin-orbit potential is

$$\begin{aligned}
J_{n,n} &= \int_0^\infty r^2 dr j_n(p_f r) V_{SO}(r) j_n(p_i r) = \int_0^\infty r^2 dr j_n(p_f r) \left[ \frac{1}{r} \frac{V_0(r)}{dr} \right] j_n(p_i r) \\
&= \frac{\pi}{2\sqrt{p_f p_i}} \int_0^\infty dr J_{n+1/2}(p_f r) \left[ \frac{V_0(r)}{dr} \right] J_{n+1/2}(p_i r) \\
&= -\frac{\pi}{2\sqrt{p_f p_i}} \int_0^\infty dr \frac{d}{dr} [J_{n+1/2}(p_f r) J_{n+1/2}(p_i r)] V_0(r) \\
&= -\frac{\pi}{4\sqrt{p_f p_i}} \int_0^\infty dr \left\{ p_i J_{n+1/2}(p_f r) [J_{n-1/2}(p_i r) - J_{n+3/2}(p_i r)] \right. \\
&\quad \left. + p_f [J_{n-1/2}(p_f r) - J_{n+3/2}(p_f r)] J_{n+1/2}(p_i r) \right\} V_0(r).
\end{aligned}$$

In passing we note that here for  $n \geq 1$ , relevant for the spin-orbit, there is no stock term in the partial integration step. Using the recurrence relation, see [9] formula (9.1.27),

$$\frac{2v}{z} J_v = J_{v-1}(z) + J_{v+1}(z),$$

we get

$$\begin{aligned}
J_{n,n} &= -\frac{\pi}{4} \sqrt{p_f p_i} (2n+1)^{-1} \int_0^\infty r dr \left\{ [J_{n-1/2}(p_f r) + J_{n+3/2}(p_f r)] [J_{n-1/2}(p_i r) - J_{n+3/2}(p_i r)] \right. \\
&\quad \left. + [J_{n-1/2}(p_f r) - J_{n+3/2}(p_f r)] [J_{n-1/2}(p_i r) + J_{n+3/2}(p_i r)] \right\} \\
&= -\frac{\pi}{2} \sqrt{p_f p_i} (2n+1)^{-1} \int_0^\infty r dr \left\{ J_{n-1/2}(p_f r) J_{n-1/2}(p_i r) - J_{n+3/2}(p_f r) J_{n+3/2}(p_i r) \right\} V_0(r) \\
&\equiv -(p_f p_i) \cdot (2n+1)^{-1} [I_{n-1/2, n-1/2} - I_{n+3/2, n+3/2}].
\end{aligned} \tag{4.20}$$

This partial wave integral for the spin-orbit is in accordance with [10], formula (43)-(44).

Now, from  $\mathbf{q} = (\mathbf{p}' + \mathbf{p})/2$  it follows that

$$\left[ \int d^3 r e^{+i\mathbf{p}' \cdot \mathbf{r}} \mathbf{S} \cdot (\mathbf{r} \times \mathbf{q}) V_{SO}(r) e^{-i\mathbf{p} \cdot \mathbf{r}} \right] = \left[ \int d^3 r e^{+i\mathbf{p}' \cdot \mathbf{r}} \mathbf{L} \cdot \mathbf{S} V_{SO}(r) e^{-i\mathbf{p} \cdot \mathbf{r}} \right].$$

Then, with application of the Bauer formula (4.10) etc. one arrives at

$$\begin{aligned}
\langle p_f L_f m_f | \frac{i}{2} (\sigma_1 + \sigma_2) \cdot \mathbf{n} \tilde{V}_{SO}(\mathbf{k}^2) | p_i L_i m_i \rangle &= -(4\pi)^2 i^{L_f - L_i} \\
&\times \int d^3 r [Y_{L_f m_f}^*(\hat{\mathbf{r}}) \mathbf{L} \cdot \mathbf{S} Y_{L_i m_i}(\hat{\mathbf{r}})] j_{L_f}(p_f r) V_{SO}(r) j_{L_i}(p_i r) \Rightarrow -(4\pi)^2 \delta_{L_f, L_i} \delta_{M_f, M_i} \cdot \\
&\times (L_f S_f; J M_f || \mathbf{L} \cdot \mathbf{S} || L_i S_i; J M_i) \int_0^\infty r^2 dr j_{L_f}(p_f r) V_{SO}(r) j_{L_i}(p_i r).
\end{aligned} \tag{4.21}$$

Here, we incorporated the spin  $S$  and the total angular momentum  $J$ , and

$$\int d\Omega_{\hat{\mathbf{r}}} [\mathcal{Y}_{L_f S_f, J M_f}^*(\hat{\mathbf{r}}) (\mathbf{L} \cdot \mathbf{S}) \mathcal{Y}_{L_i S_i, J M_i}(\hat{\mathbf{r}})] \equiv (L_f S_f, J || \mathbf{L} \cdot \mathbf{S} || L_i S_i, J) \delta_{M_f, M_i}, \tag{4.22}$$

with for total spin  $S=1$  and angular momentum  $J$ , the matrix elements are non-zero. We have two cases:

(i) triplet-uncoupled:  $L_f = L_i = J$ :  $\langle \mathbf{L} \cdot \mathbf{S} \rangle = (J || \mathbf{L} \cdot \mathbf{S} || J) = -1$ .

(ii) triplet-coupled:  $L_f = J \pm 1$  and  $L_i = J \pm 1$ :  $L_f$  and  $L_i$  are  $L \pm 1$ , the spin-orbit is diagonal in  $L$ :

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = (L_f S_f J || \mathbf{L} \cdot \mathbf{S} || L_i S_i J) = \begin{pmatrix} J-1 & 0 \\ 0 & -(J+2) \end{pmatrix} \tag{4.23}$$

### E. Fourier-Transform Quadratic-Spin-Orbit-Potential

The partial-wave matrix elements in momentum space can be related to those in configuration space as follows

$$\begin{aligned}
& \langle p_f L_f m_f | \left( \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}) \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}) \right) \tilde{V}_Q(\mathbf{k}^2) | p_i L_i m_i \rangle = \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \cdot \\
& \times \langle p_f L_f m_f | b f p' \rangle \langle p' | \left( \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}) \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}) \right) \tilde{V}_Q(\mathbf{k}^2) | p \rangle \langle p | p_i L_i m_i \rangle \Rightarrow \\
& - \int d\Omega_{\hat{p}'} \int d\Omega_{\hat{p}} Y_{L_f m_f}^*(\hat{p}') Y_{L_i m_i}(\hat{p}) \cdot \\
& \times \left[ \int d^3 r e^{+i\mathbf{k} \cdot \mathbf{r}} \left( \sigma_1 \cdot (\mathbf{q} \times \nabla) \sigma_2 \cdot (\mathbf{q} \times \nabla) \right) V_0(r) \right].
\end{aligned} \tag{4.24}$$

Now, using for  $F = F(r)$  the identity

$$\nabla_l \nabla_n F(r) = \frac{1}{r} F' \delta_{ln} + \left( F'' - \frac{1}{r} F' \right) \frac{x_l x_n}{r^2},$$

one gets

$$\begin{aligned}
& \sigma_1 \cdot (\mathbf{q} \times \nabla) \sigma_2 \cdot (\mathbf{q} \times \nabla) = \left[ \sigma_1 \cdot \sigma_2 \mathbf{q}^2 - \sigma_1 \cdot \mathbf{q} \sigma_2 \cdot \mathbf{q} \right] \frac{1}{r} F'(r) \\
& + \frac{1}{2} \left( (\sigma_1 \cdot \mathbf{q} \times \mathbf{r}) (\sigma_2 \cdot \mathbf{q} \times \mathbf{r}) + (\sigma_2 \cdot \mathbf{q} \times \mathbf{r}) (\sigma_1 \cdot \mathbf{q} \times \mathbf{r}) \right) \cdot \frac{1}{r^2} \left[ F'' - \frac{1}{r} F' \right].
\end{aligned}$$

Neglecting the first non-local term, we arrive at

$$\begin{aligned}
& \langle p_f L_f m_f | \left( \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}) \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}) \right) \tilde{V}_Q(\mathbf{k}^2) | p_i L_i m_i \rangle \\
& \approx - \int d\Omega_{\hat{p}'} \int d\Omega_{\hat{p}} Y_{L_f m_f}^*(\hat{p}') Y_{L_i m_i}(\hat{p}) \cdot \left[ \int d^3 r e^{-i\mathbf{p}' \cdot \mathbf{r}} V_Q(r) Q_{12}(\hat{\mathbf{r}}) e^{+i\mathbf{p} \cdot \mathbf{r}} \right].
\end{aligned} \tag{4.25}$$

Here,

$$\begin{aligned}
Q_{12}(\hat{\mathbf{r}}) &= \frac{1}{2} \left[ (\sigma_1 \cdot \mathbf{L})(\sigma_2 \cdot \mathbf{L}) + (\sigma_2 \cdot \mathbf{L})(\sigma_1 \cdot \mathbf{L}) \right], \\
&= 2(\mathbf{L} \cdot \mathbf{S})^2 + \mathbf{L} \cdot \mathbf{S} - \mathbf{L}^2,
\end{aligned} \tag{4.26}$$

$$V_Q(r) = -\frac{3}{r^2} V_T(r) = -\frac{1}{r^2} \left( \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) V_0(r). \tag{4.27}$$

Substitution into (4.25) leads again to the desired formula

$$\begin{aligned}
& \langle p_f L_f m_f | \left[ \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}) \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}) \right] \tilde{V}_Q(\mathbf{k}^2) | p_i L_i m_i \rangle = \\
& + (4\pi)^2 i^{L_f - L_i} \int d^3 r \left[ Y_{L_f m_f}^*(\hat{\mathbf{r}}) Q_{12}(\hat{\mathbf{r}}) Y_{L_i m_i}(\hat{\mathbf{r}}) \right] V_Q(r) \Rightarrow \\
& + (4\pi)^2 i^{L_f - L_i} \delta_{m_f, m_i} (L_f \| Q_{12} \| L_i) \int_0^\infty r^2 dr j_{L_f}(p_f r) V_Q(r) j_{L_i}(p_i r).
\end{aligned} \tag{4.28}$$

Here,

$$\int d\Omega_{\hat{\mathbf{r}}} \left[ Y_{L_f m_f}^*(\hat{\mathbf{r}}) Q_{12}(\hat{\mathbf{r}}) Y_{L_i m_i}(\hat{\mathbf{r}}) \right] \equiv (L_f \| Q_{12} \| L_i) \delta_{m_f, m_i}, \tag{4.29}$$

We have three cases:

(i) spin singlet:  $L_f = L_i = J$ :  $\langle Q_{12} \rangle = (J \| Q_{12} \| J) = -J(J+1)$ .

(ii) spin triplet-uncoupled:  $L_f = L_i = J$ :  $\langle Q_{12} \rangle = (J \| Q_{12} \| J) = 1 - J(J+1)$ .

(ii) spin triplet-coupled:  $L_f = J \pm 1$  and  $L_i = J \pm 1$ :  $L_f$  and  $L_i$  are  $L \pm 1$ ,

$$\langle Q_{12} \rangle = (L_f \| Q_{12} \| L_i) = \begin{pmatrix} (J-1)^2 & 0 \\ 0 & (J+2)^2 \end{pmatrix} \quad (4.30)$$

### F. Non-local Potential

We note that the  $\mathbf{q}^2$ -terms contain a non-local and a local part as seen from

$$\mathbf{q}^2 = \left( \mathbf{q}^2 + \frac{1}{4} \mathbf{k}^2 \right) - \mathbf{k}^2/4. \quad (4.31)$$

We note that the second term is contained in the local part of the configuration space potential. Therefore, only the first term (...) has to be included in addition to the local potentials, which corresponds to the  $\phi_C$  and  $\phi_\sigma$  functions.

(i) **Central Potential:** The non-local factor  $\mathbf{q}^2 + \mathbf{k}^2/4 = (p_f^2 + p_i^2)/2$  is angle independent and therefore the partial wave expansion is very similar to that for the local potentials. Analogous to (4.11), we have

$$\begin{aligned} \langle p_f L_f m_f | (\mathbf{q}^2 + \mathbf{k}^2/4) \tilde{V}_{nl}(\mathbf{k}^2) | p_i L_i m_i \rangle &= +(4\pi)^2 \delta_{L_f, L_i} \delta_{S_f, S_i} \delta_{m_f, m_i} \cdot \\ &\times \frac{1}{2} (p_f^2 + p_i^2) \int_0^\infty r^2 dr j_{L_f}(p_f r) V_{nl}(r) j_{L_i}(p_i r), \end{aligned} \quad (4.32)$$

where

$$V_{nl}(r) = [\phi_C(r) + (4S-3) \phi_\sigma(r)] / (2m_{red}), \quad (4.33)$$

with  $S_f = S_i = S$ , and  $4S-3 = 2S(S+1) - 3$  for  $S=0,1$ .

(ii) **Tensor Potential:** In ESC-models pseudoscalar exchange include the so-called 'Graz-correction'

$$\begin{aligned} \Delta \tilde{V}_{PS}^{graz} &= \frac{f_{NN\pi}^2}{m_\pi^2} \frac{1}{2M_N^2} (\mathbf{q}^2 + \mathbf{k}^2/4) \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k}}{\mathbf{k}^2 + m^2} \exp(-\mathbf{k}^2/\Lambda^2) \\ &\equiv (\mathbf{q}^2 + \mathbf{k}^2/4) (\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) \tilde{V}_{nl,t}(\mathbf{k}^2). \end{aligned} \quad (4.34)$$

Separation of the spin-spin and the tensor part we get

1. **Spin-spin:** As in (4.32)

$$\begin{aligned} \frac{1}{3}(4S-3) \langle p_f L_f m_f | (\mathbf{q}^2 + \mathbf{k}^2/4) \mathbf{k}^2 \tilde{V}_{nl,t}(\mathbf{k}^2) | p_i L_i m_i \rangle &= +(4\pi)^2 \delta_{L_f, L_i} \delta_{S_f, S_i} \delta_{m_f, m_i} \cdot \\ &\times -\frac{1}{6}(4S-3) (p_f^2 + p_i^2) \int_0^\infty r^2 dr j_{L_f}(p_f r) \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) V_{nl,t}(r) j_{L_i}(p_i r). \end{aligned} \quad (4.35)$$

In the soft-core potentials we have that  $V_{nl,t} \propto \phi_C^0(r)$ , which implies that  $-\nabla^2 V_{nl,t} \propto m^2 \phi_C^1(r)$ .

2. **Tensor:** Now the matrix element is

$$\begin{aligned} \langle p_f L_f m_f | (\mathbf{q}^2 + \mathbf{k}^2/4) \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_{nl,t}(\mathbf{k}^2) | p_i L_i m_i \rangle &= \\ \frac{1}{2} (p_f^2 + p_i^2) \langle p_f L_f m_f | \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_{nl,t}(\mathbf{k}^2) | p_i L_i m_i \rangle, \end{aligned} \quad (4.36)$$

which gives, using the result (4.15),

$$\begin{aligned} \langle p_f L_f m_f | (\mathbf{q}^2 + \mathbf{k}^2/4) \left( \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - \frac{1}{3} \mathbf{k}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) \tilde{V}_{nl,t}(\mathbf{k}^2) | p_i L_i m_i \rangle &= \\ -(4\pi)^2 i^{L_f - L_i} \delta_{M_f, M_i} (J M_f, S_f L_f \| S_{12} \| J M_i, L_i S_i) \cdot \frac{1}{2} (p_f^2 + p_i^2) \cdot \\ &\times \int_0^\infty r^2 dr \left[ j_{L_f}(p_f r) \frac{1}{3} \left( \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) V_{nl,t}(r) j_{L_i}(p_i r) \right]. \end{aligned} \quad (4.37)$$



3. **Integral equation for  $\widetilde{KJ}$ :** With an extra (non-local) factor  $\mathbf{q}^2 + \mathbf{k}^2/4 = (\mathbf{p}_f^2 + \mathbf{p}_i^2)/2$  the integral-equation does not become a non-fredholm integral equation. The Gaussian form factor gives enough damping for the off-shell potential matrix elements for guaranteeing Fredholm properties! (See e.g. thesis P. Verhoeven [11]).

## V. YUKAWIAN POTENTIALS WITH CUT-OFF

Gaussian expansion: The Feynman/Yukawa propagator in the Schwinger parametrization

$$\begin{aligned}\widetilde{V}(\mathbf{k}^2) &= \widetilde{g}^2 \frac{e^{-\mathbf{k}^2/\Lambda^2}}{\mathbf{k}^2 + m^2} = \widetilde{g}^2 \int_0^\infty d\alpha e^{-\alpha(\mathbf{k}^2 + m^2)} e^{-\mathbf{k}^2/\Lambda^2} \\ &= \widetilde{g}^2 \exp\left(\frac{m^2}{\Lambda^2}\right) \int_{1/\Lambda^2}^\infty d\gamma e^{-\gamma m^2} e^{-\gamma \mathbf{k}^2}.\end{aligned}\quad (5.1)$$

In configuration space this gives

$$\begin{aligned}V_G(r) &= \frac{\widetilde{g}^2}{8\pi\sqrt{\pi}} e^{m^2/\Lambda^2} \int_{1/\Lambda^2}^\infty \frac{d\gamma}{\gamma\sqrt{\gamma}} e^{-r^2/(4\gamma)} \text{ with } \gamma = 1/4\mu^2 \rightarrow \\ &= \frac{\widetilde{g}^2}{2\pi\sqrt{\pi}} e^{m^2/\Lambda^2} \int_0^{2\Lambda} d\mu e^{-\mu^2 r^2} \approx \sum_i w_i e^{-\mu_i^2 r^2},\end{aligned}\quad (5.2)$$

i.e. a quadrature approximation as a sum of gaussians, with  $0 < \mu_i < 2\Lambda$ .

*It appears from (5.1) and (5.2) that a numerical fit for the ESC potentials with GEM (gaussians) is very natural and superior over a fit with exponentials!.*

## VI. GAUSS-BESSEL RADIAL INTEGRALS

In this section we evaluate the Double-Bessel transform of the potential  $V(r)$ . These are the radial integrals

$$I_{L',L}(a,b) = \int_0^\infty r^2 dr V(r) j_{L'}(ar) j_L(br), \quad (6.1)$$

where  $a = q_f$  and  $b = q_i$ . Using the standard numerical quadratures gives problems for large momenta, and in particular for the far off-energy-shell matrix elements  $\langle q_f, L' | V | q_i, L \rangle$ . Here we describe the method based on the expansion

$$V(r) = \sum_{k=1}^N A_k r^n \exp[-\mu_k^2 r^2], \quad (6.2)$$

for general  $L', L$ , using the closed analytical expression for the integrals (6.1) given in [12]. For the expansion of the potential in gaussians the partial-wave Namely, the basic integrals are

$$\begin{aligned}\int_0^\infty x dx e^{-\rho^2 x^2} J_p(\beta x) J_p(\gamma x) &= \frac{1}{2\rho^2} \exp\left(-\frac{\beta^2 + \gamma^2}{4\rho^2}\right) I_p\left(\frac{\beta\gamma}{2\rho^2}\right) \\ &\left[ \operatorname{Re} p > -1, |\arg p| < \frac{1}{4}, \alpha > 0, \beta > 0 \right].\end{aligned}\quad (6.3)$$

### A. Gaussian Expansion Central Potentials $V_T(r)$

Application of (6.3) gives

$$\begin{aligned}
 I_{n,n}(a, b; \mu^2) &= \int_0^\infty r^2 dr e^{-\mu^2 r^2} j_n(ar) j_n(br) \\
 &= \frac{\pi}{2\sqrt{ab}} \int_0^\infty r dr e^{-\mu^2 r^2} J_{n+\frac{1}{2}}(ar) J_{n+\frac{1}{2}}(br) \\
 &= \frac{\pi}{4\mu^2 \sqrt{ab}} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) I_{n+1/2}\left(\frac{ab}{2\mu^2}\right) \\
 &= \frac{\sqrt{\pi}}{4\mu^3} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) f_n\left(\frac{ab}{2\mu^2}\right),
 \end{aligned} \tag{6.4}$$

where  $f_n(X) = \sqrt{\pi/2X} I_{n+1/2}(X)$ , see [9], section (10.2). The recurrence relations read, [9] 10.2.18-10.2.20,

$$\begin{aligned}
 f_{n-1}(X) - f_{n+1}(X) &= (2n+1) f_n(X)/X, \\
 n f_{n-1}(X) + (n+1) f_{n+1}(X) &= (2n+1) f'_n(X), \\
 (n+1) f_n(X)/X + f'_n(X) &= f_{n-1}(X), \\
 -n f_n(X)/X + f'_n(X) &= f_{n+1}(X).
 \end{aligned} \tag{6.5}$$

### B. Gaussian Expansion Tensor potentials $V_T(r)$ (I)

The tensor-potential in configuration space is given by [13]

$$V_T(r) = -(1/6\pi^2) \int_0^\infty dk k^4 j_2(kr) V_3(k^2) \rightarrow r^2 \text{ for } r \rightarrow 0. \tag{6.6}$$

In the case of the soft-core potentials the integral in (6.6) exists for all  $0 < r < \infty$ .

Now,

$$V_T(r) = \frac{1}{3} \left( \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) V_0(r),$$

and making the GEM expansion

$$V_0(r) = \sum_{k=1}^N a_k \exp[-\mu_k^2 r^2],$$

we have for the tensor potential the logical expansion

$$V_T(r) = \frac{4}{3} \sum_{k=1}^N (a_k \mu_k^2) (\mu_k r)^2 \exp[-\mu_k^2 r^2].$$

1. For the diagonal matrix elements  $\langle L || V_T || L \rangle$  we have the following integral ( $L=n$ ):

$$J_{n,n}^{(2)}(a, b; \mu^2) = \int_0^\infty r^2 dr \left[ r^2 e^{-\mu^2 r^2} \right] j_n(ar) j_n(br) = \left( -\frac{d}{d\mu^2} \right) I_{n,n}(a, b; \mu^2) \tag{6.7}$$

Performing the derivative gives

$$\begin{aligned}
 J_{n,n}^{(2)}(a, b; \mu^2) &= \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \left(3 - \frac{a^2 + b^2}{2\mu^2}\right) f_n(X) + 2X f'_n(X) \right] \\
 &= -\frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \frac{a^2 + b^2}{2\mu^2} f_n(X) \right. \\
 &\quad \left. - X \left( \frac{2n+3}{2n+1} f_{n-1}(X) + \frac{2n-1}{2n+1} f_{n+1}(X) \right) \right],
 \end{aligned} \tag{6.8}$$

where  $X = (ab)/(2\mu^2)$ . Notice that the expression (6.8) is in complete analogy with the expressions for  $V_{1,1}^J(P)$  and  $V_{3,3}^J(P)$ , with  $n = J - 1$  and  $n = J + 1$  respectively, of [10] on p. 13. The connection is given by the substitutions

$$\sqrt{(q_f^2 + q_i^2)} \sin \psi = q_f \rightarrow \frac{a}{\mu\sqrt{2}}, \quad \sqrt{(q_f^2 + q_i^2)} \cos \psi = q_i \rightarrow \frac{b}{\mu\sqrt{2}}, \quad V_J^{(T)} \rightarrow f_n(X), \quad (6.9)$$

and

$$\frac{8\pi}{3} \rightarrow \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right).$$

2. For the non-diagonal matrix elements  $\langle L || V_T || L + 2 \rangle$  we first consider the following integral ( $L=n$ ):

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &= \int_0^\infty r^2 dr \left[ r^2 e^{-\mu^2 r^2} \right] j_n(ar) j_{n+2}(br) \\ &= \frac{\pi}{2\sqrt{ab}} \int_0^\infty r^3 dr e^{-\mu^2 r^2} J_{n+\frac{1}{2}}(ar) J_{n+\frac{5}{2}}(br). \end{aligned} \quad (6.10)$$

Now,

$$J_{n+5/2}(br) = (2n+3) \left[ \frac{(2n+1)}{2b^2 r^2} - \frac{1}{r^2} \frac{1}{b} \frac{d}{db} \right] J_{n+1/2}(br) - J_{n+1/2}(br), \quad (6.11)$$

which gives in (6.10)

$$\begin{aligned} \frac{2\sqrt{ab}}{\pi} J_{n,n+2}^{(2)}(a, b; \mu^2) &= (2n+3) \left[ \frac{(2n+1)}{2b^2} - \frac{1}{b} \frac{d}{db} \right] \int_0^\infty r dr e^{-\mu^2 r^2} J_{n+\frac{1}{2}}(ar) J_{n+\frac{1}{2}}(br) \\ &\quad + \left( \frac{d}{d\mu^2} \right) \int_0^\infty r dr e^{-\mu^2 r^2} J_{n+\frac{1}{2}}(ar) J_{n+\frac{1}{2}}(br) \end{aligned}$$

leading, using

$$\frac{1}{\sqrt{b}} \frac{1}{b db} = \left( \frac{1}{2b^2} + \frac{1}{b db} \right) \frac{1}{\sqrt{b}},$$

to the expression

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &= \left\{ (2n+3) \left[ \frac{2n}{2b^2} - \frac{1}{b} \frac{d}{db} \right] + \frac{d}{d\mu^2} \right\} I_{n,n}(a, b; \mu^2) \\ &= -J_{n,n}^{(2)}(a, b; \mu^2) + \frac{\sqrt{\pi}(2n+3)}{8(\mu^3 b^2)} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \\ &\quad \times \left[ \left( 2n + \frac{b^2}{\mu^2} \right) f_n(X) - \frac{ab}{\mu^2} f'_n(X) \right]. \end{aligned} \quad (6.12)$$

Now, using the recurrences (6.5),

$$\begin{aligned} \frac{\mu^2}{b^2} \left( 2nf_n(X) - \frac{ab}{\mu^2} f'_n(X) \right) &= \frac{\mu^2}{b^2} \left( 2X f'_n(X) - 2X f_{n+1}(X) - 2X f'_n(X) \right) = \\ &= -\frac{\mu^2}{b^2} \cdot \frac{2X^2}{n+2} \left( f_n(X) - f'_{n+1}(X) \right) = -\frac{a^2}{2\mu^2} \frac{1}{n+2} \left( f_n(X) - f'_{n+1}(X) \right), \end{aligned}$$

which leads to the expression

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &= -J_{n,n}^{(2)}(a, b; \mu^2) + \frac{\sqrt{\pi}}{8\mu^5} (2n+3) \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \\ &\quad \times \left[ f_n(X) - \frac{a^2}{2\mu^2} \frac{1}{n+2} \left( f_n(X) - f'_{n+1}(X) \right) \right] \\ &= \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \left( \frac{a^2 + b^2}{2\mu^2} - \frac{2n+3}{n+2} \frac{a^2}{2\mu^2} \right) f_n(X) \right. \\ &\quad \left. - \frac{ab}{\mu^2} f_{n+1}(X) + \frac{2n+3}{n+2} \frac{a^2}{2\mu^2} f'_{n+1}(X) \right]. \end{aligned} \quad (6.13)$$

where again  $X = (ab)/(2\mu^2)$ . Now, using  $f'_{n+1} = f_n - (n+2)f_{n+1}/X$  we get for the expression [...] in (6.13)

$$\begin{aligned} \left[ \dots \right] &= \frac{a^2 + b^2}{2\mu^2} f_n - \frac{ab}{\mu^2} f_{n+1} - (2n+3) \frac{a^2}{2\mu^2} \frac{1}{X} f_{n+1} = \\ &= \frac{b^2}{2\mu^2} f_n - \frac{ab}{\mu^2} f_{n+1} + \frac{a^2}{2\mu^2} f_{n+2}, \end{aligned}$$

and

$$J_{n,n+2}^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \frac{b^2}{2\mu^2} f_n(X) - \frac{ab}{\mu^2} f_{n+1}(X) + \frac{a^2}{2\mu^2} f_{n+2}(X) \right]. \quad (6.14)$$

The analog with the expression for  $V_{3,1}^I(P)$  of [10], eqn. (55), is, using the substitutions (6.9),

$$J_{j-1,j+1}^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \frac{ab}{\mu^2} f_j(X) - \left( \frac{b^2}{2\mu^2} f_{j-1}(X) + \frac{a^2}{2\mu^2} f_{j+1}(X) \right) \right]. \quad (6.15)$$

So, apart from a (-)-sign, which is included elsewhere, (6.14) and (6.15) are in agreement for  $n = j - 1$ . compare with [10], Eqn.'s (38)-(41).

The low momentum behavior of the r.h.s. in (6.14) is given by  $f_n(X) \sim X^n/(2n+1)!!$ , and we get

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &\sim \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \cdot \frac{b^2}{2\mu^2} \frac{X^n}{(2n+5)!!} \cdot \\ &\times \left[ (2n+5)(2n+3) - (2n+5) \frac{a^2}{\mu^2} + \frac{a^4}{4\mu^4} \right] + \dots \end{aligned} \quad (6.16)$$

## VII. APPLICATION TO ESC-POTENTIALS

### A. Set of Gaussians with geometric progression

Analogous to the works of Kamimura, Hiyama, and Kino [1] we expand the potentials  $V_i(r)$ , with  $i = C, \sigma, T, SO, SO_2$  etc.

$$\begin{aligned} V_i(r) &= \sum_{n=1}^{n_{max}} a_{i,nl} \phi_{n,l}^G(r), \quad \phi_{n,l}^G(r) = N_{nl} r^l e^{-v_n r^2}, \quad \text{with} \\ N_{nl} &= \left( \frac{2^{l+2} (2v_n)^{l+3/2}}{\sqrt{\pi} (2l+1)!!} \right)^{1/2}, \quad (n = 1 - n_{max}), \end{aligned} \quad (7.1)$$

where the constant  $N_{nl}$  normalizes the functions, i.e.  $\langle \phi_{n,l}^G | \phi_{n,l}^G \rangle = 1$ . It is shown by Kamimura, Hiyama, and Kino [1] that an expansion with high accuracy can be realized using a set of Gaussian range parameters in geometric progression as follows:

$$v_n = \frac{1}{r_n^2}, \quad r_n = r_1 a^{n-1} \quad (n = 1 - n_{max}). \quad (7.2)$$

In the application to the ESC-potentials we use for the central, spin-spin, spin-orbit  $l=0$ . For the tensor potentials we use  $l=2$ . The mass parameters in  $\phi_{n,l}^G$ , defined as  $\mu_n = \sqrt{v_n}$ , are displayed in Table I. The maximum mass  $\mu_{max} = 10 \hbar c \approx 2\Lambda$ . The radii  $r_n$  ( $n = 1, n_{max}$ ) in (7.2) are given in Table II.

### B. Results for ESC08c-model

(i) For each potential type ( $i = C, \sigma, T, SO, ASO, Q$ ) the fitting of the GEM coefficients consists of minimizing the  $\chi^2$ , which is defined as

$$\chi^2(\mathbf{a}_i) = \sum_{k=1}^N \left( \frac{\sum_{n=1}^{n_{max}} a_{i,n} \phi_{n,l}^G(r_k) - V_i(r_k)}{\sigma(k)} \right)^2, \quad (7.3)$$

TABLE I: GEM mass parameters in MeV, with  $n_{\max} = 30$ ,  $\mu_{\min} = \hbar c/10$ , and  $\mu_{\max} = 10 \hbar c$ .

19.733	23.129	27.109	31.775	37.244
43.653	51.166	59.972	70.293	82.391
96.571	113.191	132.671	155.504	182.267
213.635	250.402	293.497	344.009	403.213
472.607	553.944	649.279	761.021	891.995
1045.509	1225.444	1436.346	1683.544	1973.286

TABLE II: GEM  $r[\text{fm}]$  parameters in MeV, with  $n_{\max} = 30$ ,  $r_1 = \frac{\hbar}{\mu_{\max} c}$  fm,  $r_{\max} = (\hbar c/\mu_{\min})$  fm.

10.000	8.532	7.279	6.210	5.298
4.520	3.857	3.290	2.807	2.395
2.043	1.743	1.487	1.269	1.083
0.924	0.788	0.672	0.574	0.489
0.418	0.356	0.304	0.259	0.221
0.189	0.161	0.137	0.117	0.100

w.r.t. variations of the coefficients  $a_{i,n}$ , where we choose for the errors  $\sigma(k) = 1$  MeV. Here, the base functions are the Gaussians  $\phi_{n,l}^G(r) = r^l \exp(-\mu_n^2 r^2)$ , where the  $\mu_n$  are given in Table I. The radii  $r_k$  ( $k = 1, N$ ) in (7.3) are chosen to be an equidistant set of distances in the interval  $0 < r_k < 15$  fm, where for example  $N=400$ . So, they are a different set as those in (7.2).

Starting from an initial set parameters  $\mathbf{a}_i^{(0)}$  the  $\chi^2(\mathbf{a}_i)$  is developed up to second order around the initial values and the optimal values are given by minimizing the  $\chi^2$ , i.e. the solution of the equation

$$\frac{\partial \chi^2(\mathbf{a}_i)}{\partial \Delta a_{i,n}} = 0 = \frac{\partial \chi^2}{\partial a_{i,n}} + \sum_{m=1}^{n_{\max}} \frac{\partial^2 \chi^2}{\partial a_{i,n} \partial a_{i,m}} \Delta a_{i,m}. \quad (7.4)$$

Equation (7.4) is solved for  $\Delta a_{i,n}$  and via iteration the minimum is approached. Since the  $\chi^2$  is quadratic in the parameters  $a_{i,n}$  ( $n = 1, n_{\max}$ ), equation (7.4) is linear in the parameters. This makes the procedure very fast and in a few steps the minimum is reached in practice.

(ii) The numerical solution of the partial-wave Lipmann-Schwinger equation is done using either the Kowalski-Noyes [5] or the Haftel-Tabakin [6] method. The momentum integral over the interval  $(0, \infty)$  in the Lippmann-Schwinger equation is transformed to an integral over interval  $(-1, +1)$  in the variable  $y$

$$\int_0^\infty dp f(p) = \int_{-1}^{+1} dy g(y) \quad (7.5)$$

by the hyperbolic mapping

$$y = \frac{p - p_0}{p + p_0}, \quad g(y) = \frac{2p_0}{(1-y)^2} f\left(\frac{1+y}{1-y} p_0\right). \quad (7.6)$$

We use the Gauss quadrature applying it for the interval  $(-1 < y \leq 0)$  and  $(0 < y \leq +1)$ . It appeared that 40 points are adequate with  $p_0 = 1200$  MeV. The results are rather insensitive to the precise value of  $800 < p_0 < 1600$  MeV. Also, the arctangent mapping gives equivalent results.

In Table VI and Table VIII we display the ESC08c phase shifts for the computations in configuration- and momentum-space respectively. The differences are shown in Fig's 1 and 2. The solid curves represent the configuration-space and the dashed curves the momentum-space results. The agreement is satisfactory. The phases for the higher partial waves ( $L \geq 3$ ) show for  $T_{lab} \leq 5$  MeV sign changes in the momentum space computations, which seems to indicate that the fitting of the very long range parts of the potentials should be improved.

TABLE III: Meson parameters of the fitted ESC-model. Coupling constants are at  $\mathbf{k}^2 = 0$ . An asterisk denotes that the coupling constant is not searched, but constrained via SU(3) or simply put to some value used in previous work.

meson	mass (MeV)	$g/\sqrt{4\pi}$	$f/\sqrt{4\pi}$	$\Lambda$ (MeV)
$\pi$	138.04		0.2689	948.10
$\eta$	547.45		0.1142*	„
$\eta'$	957.75		0.1264*	942.74
$\rho$	768.10	0.7323	3.7754	688.75
$\phi$	1019.41	-1.2246*	2.4639*	„
$\omega$	781.95	3.5574	-0.6096	1124.26
$a_0$	982.70	0.8353		1137.66
$f_0$	974.10	-1.3072		„
$\varepsilon$	760.00	4.3553		1057.64
$a_1$	1270.00	-1.1983	0.9013	1203.56
$f_1$	1420.00	0.8153	-1.8968	„
$f'_1$	1285.00	-0.8672	1.7293	„
$b_1$	1235.00	-0.2039		948.10
$h_1$	1380.00	-0.0622		948.10
$h'_1$	1170.00	-0.0344		948.10
Pomeron	223.20	3.6832		
Odderon	273.91	3.7111	-4.5900	

TABLE IV: Pair-meson coupling constants employed in the MPE-potentials. Coupling constants are at  $\mathbf{k}^2 = 0$ . An asterisk denotes that the coupling constant is set to zero.

$J^{PC}$	SU(3)-irrep	$(\alpha\beta)$	$g/4\pi$	$f/4\pi$
$0^{++}$	$\{1\}$	$(\pi\pi)_0$	0*	
$0^{++}$	„	$(\sigma\sigma)$	0*	
$0^{++}$	$\{8\}_s$	$(\pi\eta)$	-0.0701	
$0^{++}$		$(\pi\eta')$	0*	
$1^{--}$	$\{8\}_a$	$(\pi\pi)_1$	0.0351	-0.3195
$1^{++}$	„	$(\pi\rho)_1$	0.9813	
$1^{++}$	„	$(\pi\sigma)$	-0.0315	
$1^{++}$	„	$(\pi P)$	0*	
$1^{+-}$	$\{8\}_s$	$(\pi\omega)$	-0.0413	

## VIII. DISCUSSION AND CONCLUDING REMARKS

In this paper we presented a very practical and accurate method for obtaining the momentum-space potentials from the configuration-space ones. The method is demonstrated by the application to the Extended-soft-core (ESC) potentials. The reproduction of the phase shifts as obtained in the configuration-space computations, e.g. up to ten significant digits, is rather difficult. But, with a little adjustment by refitting the meson parameters one can produce the same  $\chi^2_{p.d.p}$  for NN with the momentum space computation. For application to e.g. nuclei this is unnecessary.

The treatment of the tensor potential in this paper enforces the zero at  $r=0$  by using an extra factor  $r^2$  in the GEM-expansion. An alternative is to use instead a factor  $(1 - \exp[-U^2 r^2])$ , where  $U$  is such that only the short-range part of the potential is affected. The drawback is perhaps that it means the introduction of the new (non-linear) parameter  $U$ .

The extension of the method developed in this paper for the computation of the matrix elements of the potentials for spin 1/2-spin 1/2 systems to hyperon-nucleon (YN) and hyperon-hyperon (YY) systems is straightforward. The expansion coefficients become matrices in channel-space, which is obvious.

We have developed an analytical presentation of the ESC-potentials in momentum space for NN [2]. Of course, this also could be extended to YN and YY. Although by itself this is interesting but it seems that the method proposed in this paper is

TABLE V:  $\chi^2$  and  $\hat{\chi}^2$  per datum at the ten energy bins for the Nijmegen93 Partial-Wave-Analysis [14, 15].  $N_{\text{data}}$  lists the number of data within each energy bin. The bottom line gives the results for the total 0 – 350 MeV interval. The  $\chi^2$ -access for the ESC model is denoted by  $\Delta\chi^2$  and  $\Delta\hat{\chi}^2$ , respectively.

$T_{\text{lab}}$	# data	$\chi^2_0$	$\Delta\chi^2$	$\hat{\chi}^2_0$	$\Delta\hat{\chi}^2$
0.383	144	137.5549	19.1	0.960	0.132
1	68	38.0187	58.1	0.560	0.854
5	103	82.2257	9.8	0.800	0.095
10	209	257.9946	36.8	1.234	0.127
25	352	272.1971	45.5	0.773	0.129
50	572	547.6727	64.0	0.957	0.112
100	399	382.4493	20.3	0.959	0.051
150	676	673.0548	99.1	0.996	0.147
215	756	754.5248	130.5	0.998	0.173
320	954	945.3772	229.1	0.991	0.240
Total	4233	4091.122	712.2	0.948	0.164

more practical when the Schrödinger and Lippmann-Schwinger equations are employed in the configuration and momentum-space respectively. In the case of e.g. the Kadyshevsky formalism for baryon-baryon, like in pion-nucleon [16], the analytic presentation in momentum-space is preferable.

### Appendix A: Check $J_{n,n+2}$ -integral

The purpose of this appendix is to review and check the derivation given in section VI of the integral (6.14):

#### 1. The basic integral (6.4)

$$I_{n,n}(a, b; \mu^2) = \int_0^\infty r^2 dr e^{-\mu^2 r^2} j_n(ar) j_n(br) = \frac{\sqrt{\pi}}{4\mu^3} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) f_n\left(\frac{ab}{2\mu^2}\right), \quad (\text{A1})$$

where  $f_n(X) = \sqrt{\pi/2X} I_{n+1/2}(X)$ , see [9], section (10.2).

#### 2. The diagonal tensor-integral, see (6.7) and (6.8),

$$\begin{aligned} J_{n,n}^{(2)}(a, b; \mu^2) &= \int_0^\infty r^2 dr \left[ r^2 e^{-\mu^2 r^2} \right] j_n(ar) j_n(br) = \left( -\frac{d}{d\mu^2} \right) I_{n,n}(a, b; \mu^2) \\ &= \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \left( 3 - \frac{a^2 + b^2}{2\mu^2} \right) f_n(X) + 2X f'_n(X) \right] \\ &= -\frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \frac{a^2 + b^2}{2\mu^2} f_n(X) \right. \\ &\quad \left. - X \left( \frac{2n+3}{2n+1} f_{n-1}(X) + \frac{2n-1}{2n+1} f_{n+1}(X) \right) \right], \end{aligned} \quad (\text{A2})$$

where  $X = (ab)/(2\mu^2)$ .

#### 3. The off-diagonal tensor integral (6.10)

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &= \int_0^\infty r^2 dr \left[ r^2 e^{-\mu^2 r^2} \right] j_n(ar) j_{n+2}(br) \\ &\equiv -J_{n,n}^{(2)}(a, b; \mu^2) + (2n+3) H_n^{(2)}(a, b; \mu^2), \end{aligned} \quad (\text{A3})$$

TABLE VI: ESC08c nuclear-bar pp and np phases in degrees. No  $\phi_{\text{low}}$  constraint,  $\chi^2_{\text{p.d.p.}} = 1.112$ 

T <sub>lab</sub>	0.38	1	5	10	25	50	100	150	215	320
# data	144	68	103	290	352	572	399	676	756	954
$\Delta\chi^2$	19	58	98	36	46	64	20	99	130	229
$^1\text{S}_0(\text{np})$	54.56	62.01	63.49	59.75	50.52	39.85	25.41	14.94	4.19	-9.51
$^1\text{S}_0$	14.61	32.62	54.77	55.20	48.73	39.04	25.09	14.76	4.10	-9.54
$^3\text{S}_1$	159.39	147.78	118.27	102.75	80.87	63.15	43.91	31.70	20.14	6.40
$\epsilon_1$	0.03	0.11	0.68	1.18	1.82	2.13	2.43	2.81	3.43	4.59
$^3\text{P}_0$	0.02	0.14	1.62	3.84	8.86	11.87	9.76	4.95	-1.59	-11.00
$^3\text{P}_1$	-0.01	-0.08	-0.90	-2.05	-4.89	-8.27	-13.24	-17.36	-21.99	-28.20
$^1\text{P}_1$	-0.05	-0.19	-1.50	-3.10	-6.46	-9.93	-14.84	-18.94	-23.41	-28.68
$^3\text{P}_2$	0.00	0.01	0.22	0.67	2.51	5.79	10.82	13.94	16.27	17.48
$\epsilon_2$	-0.00	-0.00	-0.05	-0.20	-0.81	-1.72	-2.71	-2.97	-2.79	-2.10
$^3\text{D}_1$	-0.00	-0.01	-0.19	-0.69	-2.85	-6.56	-12.44	-16.69	-20.68	-25.05
$^3\text{D}_2$	0.00	0.01	0.22	0.84	3.67	9.02	17.40	22.29	24.97	24.71
$^1\text{D}_2$	0.00	0.00	0.04	0.16	0.68	1.69	3.77	5.69	7.66	9.40
$^3\text{D}_3$	0.00	0.00	0.00	0.00	0.02	0.24	1.24	2.51	3.94	5.42
$\epsilon_3$	0.00	0.00	0.01	0.08	0.55	1.60	3.48	4.86	6.04	7.07
$^3\text{F}_2$	0.00	0.00	0.00	0.01	0.11	0.34	0.81	1.13	1.22	0.53
$^3\text{F}_3$	-0.00	-0.00	-0.01	-0.03	-0.23	-0.67	-1.47	-2.05	-2.61	-3.40
$^1\text{F}_3$	-0.00	-0.00	-0.01	-0.06	-0.41	-1.10	-2.12	-2.78	-3.47	-4.72
$^3\text{F}_4$	0.00	0.00	0.00	0.00	0.02	0.11	0.48	1.02	1.89	3.14
$\epsilon_4$	-0.00	-0.00	-0.00	-0.00	-0.05	-0.19	-0.53	-0.83	-1.14	-1.46
$^3\text{G}_3$	-0.00	-0.00	-0.00	-0.00	-0.05	-0.26	-0.97	-1.75	-2.78	-4.13
$^3\text{G}_4$	0.00	0.00	0.00	0.01	0.17	0.70	2.10	3.49	5.13	7.41
$^1\text{G}_4$	0.00	0.00	0.00	0.00	0.04	0.15	0.41	0.67	1.06	1.72
$^3\text{G}_5$	-0.00	-0.00	-0.00	-0.00	-0.01	-0.06	-0.20	-0.33	-0.43	-0.42
$\epsilon_5$	0.00	0.00	0.00	0.00	0.04	0.20	0.71	1.24	1.88	2.72

with, see (6.12),

$$H_n^{(2)}(a, b; \mu^2) = \left[ \frac{2n}{2b^2} - \frac{1}{b} \frac{d}{db} \right] I_{n,n}(a, b; \mu^2) = \frac{\sqrt{\pi} \mu^2}{8\mu^5 b^2} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \left(2n + \frac{b^2}{\mu^2}\right) f_n(X) - 2X f'_n(X) \right]. \quad (\text{A4})$$

4. For checking (6.14) we now consider the combination

$$2n f_n(X) - 2X f'_n(X) = 2X \left[ f'_n(X) - f_{n+1}(X) \right] - 2X f'_n(X) = -2X f_{n+1}(X),$$

where we applied the fourth recurrence in (6.5). Next, we apply the first recurrence in (6.5) and obtain

$$2n f_n(X) - 2X f'_n(X) = -\frac{2X^2}{(2n+3)} \left( f_n(X) - f_{n+2}(X) \right).$$

This gives for (A4) the expression

$$H_n^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ f_n(X) - \frac{1}{2n+3} \left( f_n(X) - f_{n+2}(X) \right) \frac{a^2}{2\mu^2} \right]. \quad (\text{A5})$$



TABLE VII: ESC08c Low energy parameters: S-wave scattering lengths and effective ranges, deuteron binding energy  $E_B$ , and electric quadrupole  $Q_e$ . ESC08' is a fit with no  $\phi_{\text{low}}$  constraint. The asterisk denotes that the low-energy parameters were not searched.

	experimental data			ESC08c	ESC08c'
$a_{pp}(^1S_0)$	-7.823	$\pm$	0.010	-7.7699	-7.7705
$r_{pp}(^1S_0)$	2.794	$\pm$	0.015	2.7516*	2.7575*
$a_{np}(^1S_0)$	-23.715	$\pm$	0.015	-23.7264	-23.7178
$r_{np}(^1S_0)$	2.760	$\pm$	0.030	2.6914*	2.6961*
$a_{nn}(^1S_0)$	-16.40	$\pm$	0.42	-16.762	-15.7585
$r_{nn}(^1S_0)$	2.860	$\pm$	0.15	2.867*	2.8723*
$a_{np}(^3S_1)$	5.423	$\pm$	0.005	5.4270*	5.4260*
$r_{np}(^3S_1)$	1.761	$\pm$	0.005	1.7521*	1.7464*
$E_B$	-2.224644	$\pm$	0.000046	-2.224621	-2.224392
$Q_e$	0.286	$\pm$	0.002	0.2696*	0.2601*

TABLE VIII: ESC08c nuclear-bar pp and np phases in degrees. Computed with LSE via GEM-fit x-space potentials.

$T_{\text{lab}}$	0.38	1	5	10	25	50	100	150	215	320
# data	144	68	103	290	352	572	399	676	756	954
$^1S_0(\text{np})$	54.45	62.03	63.50	59.78	50.58	39.94	25.56	15.13	4.41	-9.24
$^1S_0$	14.70	32.76	54.87	55.30	48.84	39.19	25.29	15.07	4.41	-9.11
$^3S_1$	159.38	147.66	118.26	102.67	80.82	63.09	43.85	31.65	20.10	6.36
$\epsilon_1$	0.10	0.24	0.67	1.20	1.84	2.16	2.49	2.90	3.56	4.76
$^3P_0$	0.03	0.13	1.63	3.82	8.84	11.85	9.78	5.05	-1.43	-10.67
$^3P_1$	0.02	0.01	-0.87	-2.02	-4.86	-8.25	-13.26	-17.43	-22.17	-28.51
$^1P_1$	0.07	-0.08	-1.45	-3.13	-6.46	-9.94	-14.85	-18.92	-23.28	-28.24
$^3P_2$	0.02	0.04	0.23	0.67	2.51	5.79	10.80	13.89	16.22	17.33
$\epsilon_2$	0.00	0.01	-0.06	-0.21	-0.82	-1.72	-2.70	-2.96	-2.80	-2.15
$^3D_1$	0.02	0.10	-0.04	-0.62	-2.80	-6.50	-12.36	-16.57	-20.54	-24.86
$^3D_2$	0.01	0.02	0.05	0.84	3.70	8.98	17.33	22.20	24.81	24.45
$^1D_2$	-0.00	-0.01	0.08	0.16	0.69	1.69	3.75	5.64	7.59	9.26
$^3D_3$	-0.00	-0.02	-0.03	-0.01	0.02	0.26	1.30	2.57	4.05	5.60
$\epsilon_3$	-0.00	-0.02	-0.01	0.12	0.57	1.62	3.50	4.85	6.03	7.06
$^3F_2$	0.00	0.02	-0.02	0.01	0.10	0.34	0.81	1.13	1.23	0.54
$^3F_3$	-0.00	0.01	0.03	-0.02	-0.19	-0.60	-1.35	-1.90	-2.45	-3.22
$^1F_3$	0.00	0.05	-0.13	-0.10	-0.44	-1.12	-2.13	-2.80	-3.50	-4.79
$^3F_4$	0.00	0.01	-0.04	-0.01	0.02	0.11	0.52	1.07	1.90	3.17
$\epsilon_4$	0.00	0.00	0.02	-0.00	-0.05	-0.20	-0.53	-0.83	-1.14	-1.46
$^3G_3$	-0.00	-0.00	0.09	0.36	0.06	-0.19	-0.89	-1.71	-2.73	-4.07
$^1G_4$	-0.00	-0.00	-0.04	0.10	0.06	0.16	0.42	0.68	1.07	1.72
$^3H_4$	0.00	0.00	0.12	-0.03	0.02	0.03	0.11	0.20	0.35	0.56

Collecting terms for  $J_{n,n+2}^{(2)}$  we have

$$f_n : \frac{a^2 + b^2}{2\mu^2} + (2n+3) - \frac{a^2}{2\mu^2}, \quad f_{n+2} : + \frac{a^2}{2\mu^2},$$

$$f_{n-1} : - \left[ \frac{2n+3}{2n+1} X \right], \quad f_{n+1} : - \left[ \frac{2n-1}{2n+1} X \right].$$

TABLE IX: Born-approximation: ESC08c nuclear-bar pp and np phases in degrees. Computed with LSE via GEM-fit x-space potentials.

$T_{lab}$	0.38	1	5	10	25	50	100	150	215	320
$^3F_3$	-0.00	0.01	0.08	-0.01	-0.22	-0.69	-1.50	-2.09	-2.69	-3.60
$^1F_3$	0.00	0.05	-0.13	-0.10	-0.44	-1.14	-2.18	-2.89	-3.69	-5.29
$^3F_4$	0.00	0.01	-0.07	-0.07	0.01	0.11	0.48	1.00	1.73	2.84
$\epsilon_4$	0.00	0.00	0.01	-0.01	-0.06	-0.20	-0.53	-0.83	-1.13	-1.47
$^3G_3$	-0.00	-0.00	0.09	0.36	0.06	-0.19	-0.77	-1.36	-1.98	-2.59
$^3G_4$	0.00	0.01	0.06	-0.31	0.08	0.65	2.02	3.37	4.92	6.96
$^1G_4$	-0.00	-0.00	-0.05	0.07	0.07	0.17	0.42	0.69	1.06	1.69
$^3G_5$	-0.00	-0.00	-0.12	0.10	0.00	-0.05	-0.22	-0.39	-0.55	-0.65
$\epsilon_5$	-0.00	-0.00	-0.05	-0.03	0.05	0.22	0.72	1.25	1.87	2.68
$^3H_4$	0.00	0.00	0.12	-0.03	0.01	0.02	0.11	0.22	0.37	0.60
$^3H_5$	-0.00	-0.00	-0.05	0.08	-0.01	-0.08	-0.29	-0.52	-0.78	-1.07
$^1H_5$	0.00	0.00	0.24	-0.17	-0.03	-0.19	-0.53	-0.87	-1.19	-1.51
$^3H_6$	0.00	0.00	0.07	-0.06	0.00	-0.00	0.04	0.10	0.23	0.51
$\epsilon_6$	0.00	0.00	0.00	0.01	-0.01	-0.03	-0.12	-0.23	-0.36	-0.55

which leads to

$$J_{n,n+2}^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left\{ \dots \right\}$$

where the expression between the curly brackets, using the previous results, becomes

$$\begin{aligned} \left\{ \dots \right\} &= \left( \frac{b^2}{2\mu^2} + (2n+3) \right) f_n \\ &- X \left[ \frac{2n-1}{2n+1} f_{n+1} + \frac{2n+3}{2n+1} f_{n-1} \right] + \frac{a^2}{2\mu^2} f_{n+2}. \end{aligned}$$

Now, it is easy to derive that

$$-X \left[ \frac{2n-1}{2n+1} f_{n+1} + \frac{2n+3}{2n+1} f_{n-1} \right] = -X \left[ 2f_{n+1} + \frac{2n+3}{X} f_n \right].$$

These results lead to the formula

$$J_{n,n+2}^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \left[ \frac{b^2}{2\mu^2} f_n(X) - \frac{ab}{\mu^2} f_{n+1}(X) + \frac{a^2}{2\mu^2} f_{n+2}(X) \right], \quad (A6)$$

**which is the same expression as in (6.14).**

The low momentum behavior of the r.h.s. in (6.14) is given by  $f_n(X) \sim X^n/(2n+1)!!$ , and we get

$$\begin{aligned} J_{n,n+2}^{(2)}(a, b; \mu^2) &\sim \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2 + b^2}{4\mu^2}\right) \cdot \frac{b^2}{2\mu^2} \frac{X^n}{(2n+5)!!} \cdot \\ &\times \left[ (2n+5)(2n+3) - (2n+5) \frac{a^2}{\mu^2} + \frac{a^4}{4\mu^4} \right] + \dots \end{aligned} \quad (A7)$$

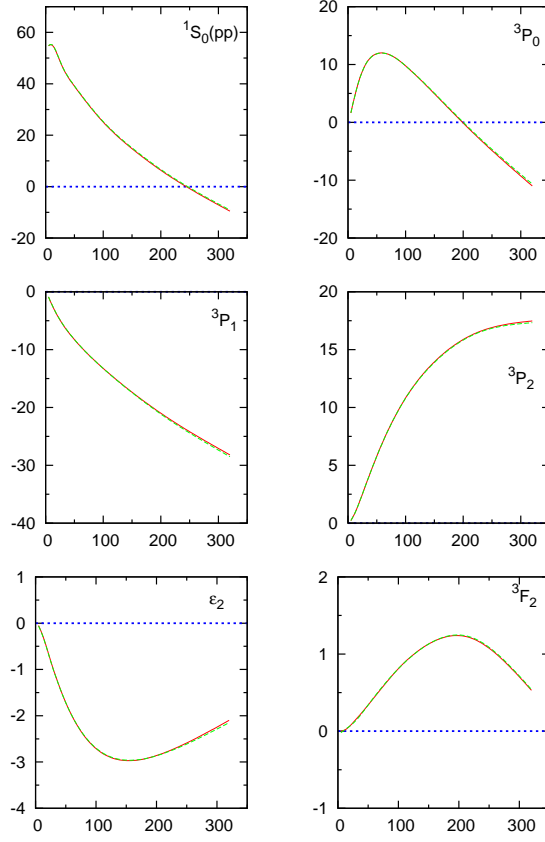


FIG. 1: Solid line: proton-proton  $I = 1$  phase shifts for the ESC08c-model. The dashed line: GEM-method momentum-space computation.

### Appendix B: Explicit evaluation $J_{0,0}^{(2)}$ - and $J_{0,2}^{(2)}$ -integral

1. For the  $I_{0,0}$ -integral we get explicitly

$$\begin{aligned}
 I_{0,0} &\equiv \int_0^\infty r^2 dr e^{-\mu^2 r^2} j_0(ar) j_0(br) = \int_0^\infty dr \left[ r^2 e^{-\mu^2 r^2} \right] \cdot \frac{\sin(ar)}{ar} \cdot \frac{\sin(br)}{br} \\
 &= -\frac{1}{4ab} \int_0^\infty dr e^{-\mu^2 r^2} (e^{iar} - e^{-iar}) (e^{ibr} - e^{-ibr}) \\
 &= -\frac{\sqrt{\pi}}{4ab\mu} \left[ e^{-(a+b)^2/4\mu^2} - e^{-(a-b)^2/4\mu^2} \right] = +\frac{\sqrt{\pi}}{2ab\mu} e^{-(a^2+b^2)/4\mu^2} \sinh\left(\frac{ab}{2\mu^2}\right) \\
 &= +\frac{\sqrt{\pi}}{4\mu^3} e^{-(a^2+b^2)/4\mu^2} f_0(X).
 \end{aligned} \tag{B1}$$

2. For the  $J_{0,2}^{(2)}$ -integral we obtain

$$\begin{aligned}
 J_{0,2}^{(2)} &\equiv \int_0^\infty r^2 dr \left[ r^2 e^{-\mu^2 r^2} \right] j_0(ar) j_2(br) \\
 &= \int_0^\infty dr \left[ r^4 e^{-\mu^2 r^2} \right] \cdot \frac{\sin(ar)}{ar} \cdot \frac{1}{br} \left( 3 \frac{\sin(br)}{(br)^2} - \sin(br) - 3 \frac{\cos(br)}{br} \right) \\
 &= -J_{0,0}^{(2)} + \frac{3}{ab^2} \int_0^\infty r dr e^{-\mu^2 r^2} \sin(ar) \left( \frac{\sin(br)}{br} - \cos(br) \right).
 \end{aligned} \tag{B2}$$

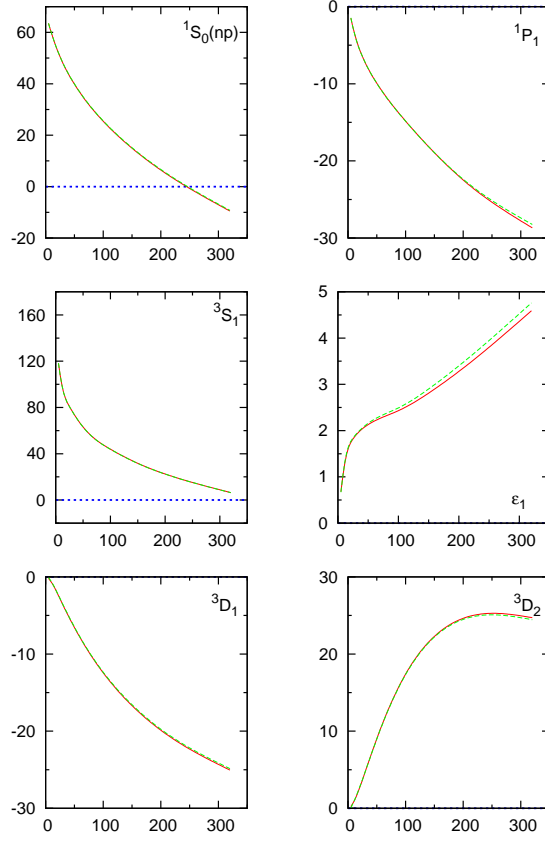


FIG. 2: Solid line: neutron-proton  $I = 0$ , and the  $I=1$   $^1S_0$ (NP) phase shifts for the ESC08c-model. The dashed line: GEM-method momentum-space computation.

3. The  $J_{0,0}^{(2)}$ -integral is defined as

$$\begin{aligned}
 J_{0,0}^{(2)} &\equiv \int_0^\infty dr \left[ r^4 e^{-\mu^2 r^2} \right] j_0(ar) j_0(br) = \frac{1}{ab} \int_0^\infty dr \left[ r^2 e^{-\mu^2 r^2} \right] \sin(ar) \sin(br) \\
 &= \frac{1}{4ab} \left( \frac{d}{d\mu^2} \right) \int_0^\infty dr e^{-\mu^2 r^2} (e^{iar} - e^{-iar}) (e^{ibr} - e^{-ibr}).
 \end{aligned} \tag{B3}$$

The explicit expression is derived as

$$\begin{aligned}
 J_{0,0}^{(2)} &= +\frac{1}{4ab} \frac{d}{d\mu^2} \left[ \frac{\sqrt{\pi}}{\mu} \left\{ e^{-(a+b)^2/4\mu^2} - e^{-(a-b)^2/4\mu^2} \right\} \right] \\
 &= -\frac{\sqrt{\pi}}{2ab} \frac{d}{d\mu^2} \left[ \frac{1}{\mu} e^{-\frac{a^2+b^2}{4\mu^2}} \sinh \frac{ab}{2\mu^2} \right] \\
 &= \frac{\sqrt{\pi}}{4ab\mu^3} e^{-\frac{a^2+b^2}{4\mu^2}} \left[ \left( 1 - \frac{a^2+b^2}{2\mu^2} \right) \sinh \left( \frac{ab}{2\mu^2} \right) + \frac{ab}{\mu^2} \cosh \left( \frac{ab}{2\mu^2} \right) \right] \\
 &= \frac{\sqrt{\pi}}{4ab\mu^3} e^{-\frac{a^2+b^2}{4\mu^2}} \cdot X \left[ \left( 1 - \frac{a^2+b^2}{2\mu^2} \right) f_0(X) + 2Xf_{-1}(X) \right] \\
 &= -\frac{\sqrt{\pi}}{8\mu^5} e^{-\frac{a^2+b^2}{4\mu^2}} \left[ \frac{a^2+b^2}{2\mu^2} f_0(X) - X(3f_{-1}(X) - f_1(X)) \right]
 \end{aligned} \tag{B4}$$

Here, in the last step we used the recurrence  $f_0 = X(f_{-1} - f_1)$ . We notice that (B4) agrees with (6.8) for  $n=0$ .

4. Writing  $J_{0,2}^{(2)} \equiv -J_{0,0}^{(2)} + H_{0,2}^{(2)}$  we have

$$\begin{aligned} H_{0,2}^{(2)} &= \frac{3}{ab^2} \int_0^\infty r dr e^{-\mu^2 r^2} \sin(ar) \left( \frac{\sin(br)}{br} - \cos(br) \right) = \\ &= \frac{3}{ab^3} \int_0^\infty dr e^{-\mu^2 r^2} \sin(ar) \sin(br) + \frac{3}{ab^2} \frac{d}{da} \int_0^\infty dr e^{-\mu^2 r^2} \cos(ar) \cos(br) \end{aligned} \quad (B5)$$

Here appear two integrals, see [9] formula **7.4.6** for the relevant integral formulas,

$$\begin{aligned} J_1 &= \int_0^\infty dr e^{-\mu^2 r^2} \sin(ar) \sin(br) = \frac{1}{2} \int_0^\infty dr e^{-\mu^2 r^2} \left\{ \cos(a-b)r - \cos(a+b)r \right\} \\ &= \frac{\sqrt{\pi}}{4\mu} \left[ \exp\left(-\frac{(a-b)^2}{4\mu^2}\right) - \exp\left(-\frac{(a+b)^2}{4\mu^2}\right) \right] \\ J_2 &= \int_0^\infty dr e^{-\mu^2 r^2} \cos(ar) \cos(br) = \frac{1}{2} \int_0^\infty dr e^{-\mu^2 r^2} \left\{ \cos(a-b)r + \cos(a+b)r \right\} \\ &= \frac{\sqrt{\pi}}{4\mu} \left[ \exp\left(-\frac{(a-b)^2}{4\mu^2}\right) + \exp\left(-\frac{(a+b)^2}{4\mu^2}\right) \right]. \end{aligned} \quad (B6)$$

$$\begin{aligned} H_{0,2}^{(2)} &= \left[ \frac{3}{ab^3} J_1 + \frac{3}{ab^2} \frac{dJ_2}{da} \right] = \frac{3\sqrt{\pi}}{2\mu ab^3} \exp\left(-\frac{a^2+b^2}{4\mu^2}\right) \left[ \sinh\left(\frac{ab}{2\mu^2}\right) \right. \\ &\quad \left. - \frac{b}{4\mu^2} \left\{ (a-b) \exp\left(\frac{ab}{2\mu^2}\right) + (a+b) \exp\left(-\frac{ab}{2\mu^2}\right) \right\} \right] \\ &= \frac{3\sqrt{\pi}}{2\mu ab^3} \exp\left(-\frac{a^2+b^2}{4\mu^2}\right) \left[ \sinh\left(\frac{ab}{2\mu^2}\right) - \frac{ab}{2\mu^2} \cosh\left(\frac{ab}{2\mu^2}\right) + \frac{b^2}{2\mu^2} \sinh\left(\frac{ab}{2\mu^2}\right) \right] \\ &= \frac{\sqrt{\pi}}{4\mu^5} \exp\left(-\frac{a^2+b^2}{4\mu^2}\right) \cdot \frac{3\mu^2}{b^2} \left[ f_0(X) - X f_{-1}(X) + \frac{b^2}{2\mu^2} f_0(X) \right]. \end{aligned} \quad (B7)$$

Useful recurrences are:

$$X f_{-1} = X f_1 + f_0, \quad f_2 = f_0 - 3f_1/X.$$

5. Collecting the results for  $J_{0,0}^{(2)}$  and  $H_{0,2}^{(2)}$  we finally arrive at

$$\begin{aligned} J_{0,2}^{(2)} &= \frac{\sqrt{\pi}}{8\mu^5} e^{-\frac{a^2+b^2}{4\mu^2}} \left[ \frac{a^2+b^2}{2\mu^2} f_0(X) - 3X f_{-1}(X) + X f_1(X) + 6\frac{\mu^2}{b^2} f_0 - 6\frac{\mu^2}{b^2} X f_{-1} + 3f_0 \right] \\ &= \frac{\sqrt{\pi}}{8\mu^5} e^{-\frac{a^2+b^2}{4\mu^2}} \left[ \frac{a^2+b^2}{2\mu^2} f_0(X) - 3X f_{-1}(X) + X f_1(X) + 6\frac{\mu^2}{b^2} f_0 - 6\frac{\mu^2}{b^2} X f_{-1} + 3f_0 \right. \\ &\quad \left. + \frac{a^2}{2\mu^2} f_2 - \frac{a^2}{2\mu^2} f_0 + \frac{3a^2}{2\mu^2 X} f_1 \right] = \frac{\sqrt{\pi}}{8\mu^5} e^{-\frac{a^2+b^2}{4\mu^2}} \cdot \left[ \frac{b^2}{2\mu^2} f_0(X) - \frac{ab}{\mu^2} f_1(X) + \frac{a^2}{2\mu^2} f_2(X) \right. \\ &\quad \left. - 3f_0(X) + 6\frac{\mu^2}{b^2} f_0(X) - 6\frac{\mu^2}{b^2} X f_1(X) - 6\frac{\mu^2}{b^2} f_0(X) + 3f_0(X) + 3\frac{a^2}{2\mu^2 X} f_1(X) \right] \\ &= \frac{\sqrt{\pi}}{8\mu^5} e^{-\frac{a^2+b^2}{4\mu^2}} \cdot \left[ \frac{b^2}{2\mu^2} f_0(X) - \frac{ab}{\mu^2} f_1(X) + \frac{a^2}{2\mu^2} f_2(X) \right]. \end{aligned} \quad (B8)$$

This result is equal to the expression in (A6) for  $n=0$ :

$$J_{0,2}^{(2)}(a, b; \mu^2) = \frac{\sqrt{\pi}}{8\mu^5} \exp\left(-\frac{a^2+b^2}{4\mu^2}\right) \left[ \frac{b^2}{2\mu^2} f_0(X) - \frac{ab}{\mu^2} f_1(X) + \frac{a^2}{2\mu^2} f_2(X) \right]. \quad (B9)$$

- [2] Th.A. Rijken, H. Polinder, and J. Nagata, Phys. Rev. C **66**, 044008 (2002); *ibid* C **66**, 044009.
- [3] For the demonstration of the method discussed in this paper we use a preliminary version of the final ESC08c-model [4].
- [4] M.M. Nagels, Th.A. Rijken, and Y. Yamamoto, *Extended-soft-core Baryon-Baryon Model ESC08. I. Nucleon-Nucleon Interactions.*, arXiv:1408.4825v1[nucl-th], 2014.
- [5] H.P. Noyes, Phys. Rev. Lett. **15** 538 (1965); K.L. Kowalski, Phys. Rev. Lett. **15** 798 (1965), erratum **15**, 908 (1965).
- [6] M.I. Haftel and F. Tabakin, Nucl. Phys. **A158** (1970) 1.
- [7] In the computer programs pspace.newrep and pspace.newrep12 we solve for the LS-equation for the  $K^{J'}$ -matrix, which is related by  $K^{J'} = 4\pi \widetilde{K^J}$  and Green function  $g' = g/(4\pi)$ . These programs are available on request.
- [8] For this formalism see e.g.: M.L. Goldberger and K.M. Watson, *Collision Theory*, New York, Wiley 1964; M.D. Scadron, *Advanced Quantum Theory*, New York, Springer-Verlag, 1979.
- [9] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc. New York, 1970.
- [10] Th.A. Rijken, R.A.M. Klomp, and J.J. de Swart, *Soft-Core OBE-Potentials in Momentum Space*, THEF-NIJM 91.05, Nijmegen preprint 1994; Published in *A Gift of Prophecy, Essays in Celebration of the Life of Robert Eugene Marshak*, Editor E.C.G. Sudarshan, World Scientific, Singapore, 1995.
- [11] P.A. Verhoeven, *Off-shell Baryon-baryon Scattering*, Ph.D.-thesis, University of Nijmegen, 1976 (unpublished).
- [12] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Academic Press, 1965.
- [13] P.M.M. Maessen, Th.A. Rijken, and J.J. de Swart, Phys. Rev. C **40** (1989) 2226.
- [14] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, Phys. Rev. C **48** (1993) 792.
- [15] R.A.M. Klomp, private communication (unpublished).
- [16] J.W. Wagenaar and T.A. Rijken, Phys. Rev. C **80**, 055204 (2009); *ibid* C **80**, 055205.